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THE PRESSURE OF A SYSTEM OF STAMPS ON AN ELASTIC HALF-PLANE UNDER GENERAL CONDITIONS OF CONTACT ADHESION AND SLIP*

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The contact interaction of an elastic half-plane and an arbitrary system of coupled and partially or completely detached stamps is considered. The problem is reduced to a combined Dirichlet-Riemann boundary value problem /1/ and is solved by quadratures. New modifications of the method and problems occurring in tasks with two and more slip sections are discussed; analogous problems with one slip section were studied earlier /2/. Fal'kovich's problem /3/ is investigated in a broadened formulation as an illustration.

1. Let $L_k = \langle a_k, b_k\rangle$, $k = 1, 2, \ldots, l$ be an open, half-open, or closed interval and $M_k = [p_k, q_k]$, $k = 1, 2, \ldots, m$, segments of the real axis $y = 0$ on which the stamps have, respectively, slipping contact and total adhesion with the elastic half-plane $-\infty < x < \infty, y \leqslant 0$; $a_1 < b_1 < \ldots < b_l, p_1 < q_1 < \ldots < q_m$. We determine the shape of the stamps, the tangential clearance on M_k, the separation-free abutment and non-intersection of the stamp and the half-plane by the boundary conditions

$$\begin{aligned} &u' = u_0'(x),\quad x \in M;\quad v' = v_0'(x),\quad x \in L \cup M; \\ &L = \bigcup_{k=1}^{l} L_k,\quad M = \bigcup_{k=1}^{m} M_k \\ &\tau_{xy} = \tau_0(x),\ x \in L;\ \sigma_y = \tau_{xy} = 0,\ x \in S;\ L \cap M = 0 \\ &\sigma_y \leqslant 0,\ x \in L;\ v(x) - v_0(x) \geqslant 0,\ x \in S' \end{aligned} \tag{1.1}$$

Here S is the **complement** $L \cup M$ to the real axis, S' are the selections outside $L \cup M$ on which the stamp base with the shape $v_0(x)$ is not contiguous to the half-plane; the given functions satisfy the Hölder condition; the interval $L_k = [a_k, b_k]$ $(L_k = (a_k, b_k))$ is

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closed (open) if sections of the free boundary S of the half-plane (the adhesion sections M_j and M_{j+1}) adjoin it from two sides; the half-open intervals $L_k = (a_k, b_k]$ or $L_k = [a_k, b_k)$ correspond to M_j being adjacent to L_k only on the left for $q_j = a_k$ or on the right for $b_k = p_j$. We give the tangential clearance $\chi_k = u(b_k) - u(a_k)$ in each open interval (a_k, b_k) We apply a normal force Y_k to each completely stripped stamp occupying the segment $[a_k, b_k]$, to each stamp having one or several adhesion sections $M_j, M_{j+1}, \ldots$ and perhaps several slip sections, one tangential force X_j' and one normal force Y_j'. The total number of parameters χ_k, Y_k, X_j', Y_j' obviously equals $l + 2m - \alpha' - 2\alpha''$, where α' is the number of half-open, and α'' the number of open intervals L_k.

We will seek the solution of the problem in the form /4/

$$\begin{aligned}&\sigma_y - i\tau_{xy} = \Phi(z) - \Phi(\bar z) + (z - \bar z)\overline{\Phi'(z)},\quad z = x + iy\\&2\mu(u' + iv') = \varkappa\Phi(z) + \Phi(\bar z) - (z - \bar z)\overline{\Phi'(z)}\\&\Phi(z) = {}^1/_4\sigma_x^\infty + 2i\mu\varepsilon^\infty(\varkappa + 1)^{-1} - Fe^{i\theta}(2\pi z)^{-1} + O(z^{-2}),\\&z \to \infty\end{aligned} \tag{1.2}$$

where F and θ are the magnitude and slope to the Ox axis of the principal vector of all the forces $Y_k, X_j', Y_j', 0 \leqslant \theta \leqslant 2\pi, \sigma_x^\infty$ is the constant component of the stress field and ε^∞ is the rotation at infinity.

Substituting (1.2) into (1.1), we obtain the combined Dirichlet-Riemann boundary value problem /1/ for a piecewise-analytic function with the boundary lines $L \cup M$

$$\begin{aligned}&\operatorname{Im}\Phi^\pm(x) = f^\pm(x),\quad f^\pm(x) = (\varkappa + 1)^{-1}[2\mu v_0'(x) \pm \tau^\pm(x)]\\&\tau^+(x) = \varkappa\tau_0(x),\ \tau^-(x) = \tau_0(x),\ x \in L\end{aligned} \tag{1.3}$$

$$\begin{aligned}&\Phi^+(x) + \varkappa\Phi^-(x) = g(x),\ g(x) = 2\mu[u_0'(x) + iv_0'(x)],\\&x \in M\end{aligned} \tag{1.4}$$

The canonical solution $X(z)$ of the homogeneous problem (1.3) and (1.4) has the form

$$X(z) = Z(z)e^{i\psi(z)}\prod_{j=1}^{l}(z - b_j)^{-\alpha_j}\prod_{j=1}^{l-1}(z - c_j)^{-\beta_j} \tag{1.5}$$

$$\begin{aligned}&Z(z) = \prod_{k=1}^{m}(z - p_k)^{-1/2+i\gamma}(z - q_k)^{-1/2-i\gamma},\quad \gamma = \frac{\ln\varkappa}{2\pi}\\&\psi(z) = \frac{1}{2\pi i}\int_L\left\{\frac{Y(z)[h^+(t) + h^-(t)]}{Y^+(t)} + h^+(t) - h^-(t)\right\}\frac{dt}{t - z}\\&Y(z) = \prod_{k=1}^{l}(z - a_k)^{1/2}(z - b_k)^{1/2},\quad Y(z) = z^l + O(z^{l-1}),\quad z \to \infty\\&Y^+(t) = i(-1)^{l-k}\left[\prod_{j=1}^{l}|t - a_j||t - b_j|\right]^{1/2},\quad t \in L_k\\&h^\pm(t) = \pi n_k^\pm - \arg Z^\pm(t) + \sum_{j=1}^{l}\alpha_j\arg(t - b_j)^\pm +\\&\sum_{j=1}^{l-1}\beta_j\arg(t - c_j)^\pm,\quad t \in L_k\end{aligned}$$

Here $n_k^\pm, \alpha_k, \beta_k \neq 0$ are integers, c_k are complex numbers, the slits in the z plane are drawn along the real axis in the positive direction, $Z(z)$ is the canonical solution of the homogeneous Riemann problem (1.4) in the broadest class of functions integrable at the nodes $p_k, q_k, k = 1, 2, \ldots, m$; $\psi(z)$ is the solution of the Dirichlet problem $\operatorname{Re}\psi^\pm(x) = h^\pm(x),\ x \in L$, bounded at the nodes $a_k, b_k, k = 1, 2, \ldots, l$ and at infinity, which is possible only when the following conditions are satisfied

$$\int_L\frac{h^+(t) + h^-(t)}{Y^+(t)}t^{j-1}dt = 0,\quad j = 1, 2, \ldots, l - 1 \tag{1.6}$$

Allowing substantial arbitrariness in the selection of the numbers β_k and c_k, the form of the solution (1.5) and (1.6) indeed generates the problem of this selection. The exception is the case $l = 1$ /2/, when the factors $(z - c_k)^{-\beta_k}$ do not occur in the function $X(z)$ independently of the quantity m.

The general solution of the homogeneous Dirichlet-Riemann problem is constructed in /1, 2/ in the form of a sum of linearly independent canonical solutions. Another method is applied below, that uses one canonical solution. Different modifications of the method enable a general solution to be obtained for a given relationship between the parameters l, m, α' and α'' in a most simple and convenient form.

The general solution of problem (1.1) and (1.2) will be sought in the broadest class of functions $\Phi(z)$ governing the finite local energy of elastic strains of a half-plane in the neighbourhood of the ends of all intervals L_k, M_j and constants at infinity. This corresponds to solving problem (1.3), (1.4) in the broadest class of piecewise-analytic functions with the boundary lines $L \cup M$ /5/. However, unlike the Dirichlet and Riemann problems, the canonical solution (1.5) and (1.6) of the combined Dirichlet-Riemann problem cannot be constructed in this class of functions in the general case.

Indeed, in the neighbourhood of the ends of L_k the asymptotic forms of the functions $X(z)$ have the form

$$X(z) = O[(z-a_k)^{\mu_k}], \quad z \to a_k; \quad X(z) = O[(z-b_k)^{\nu_k}], \quad z \to b_k \tag{1.7}$$

$$\mu_k = \delta_k + \omega_k - {}^1/_2 w_k^-, \quad \nu_k = \varepsilon_k - \omega_k + {}^1/_2 w_k^- - \alpha_k, \quad w_k^- = n_k^+ - n_k^- \tag{1.8}$$

$$\omega_k = \theta_k - \frac{1}{2\pi} \arg \frac{Z^+(x)}{Z^-(x)}\Big|_{x\in L_k}, \quad \theta_k = \sum_{j=1}^{k-1} \alpha_k \quad (k > 1), \quad \theta_1 = 0 \tag{1.9}$$

where $\delta_k = -{}^1/_2$ $(\delta_k = 0)$, if the point a_k agrees (does not agree) one some point q_j; $\varepsilon_k = -{}^1/_2$ $(\varepsilon_k = 0)$, if the point b_k agrees (does not agree) with the point p_{j+1}. Since the function $\arg\{Z^+(x)[Z^-(x)]^{-1}\}$ is constant and a multiple of 2π on L_k and α_k are integers, the numbers ω_k are also integers.

Let the function $X(z)$ have integrable singularities at both nodes of L_k. Then it follows from the form of the numbers (1.8) that $\mu_k = \nu_k = -{}^1/_2$. Combining Eqs.(1.8), we obtain the relationship $\alpha_k = \delta_k + \varepsilon_k + 1$, by virtue of which the numbers α_k can be integers only for $\delta_k = \varepsilon_k$. If $\delta_k = \varepsilon_k = -{}^1/_2$ $(L_k = (a_k, b_k))$, then $\alpha_k = 0$, if $\delta_k = \varepsilon_k = 0$ $(L_k = [a_k, b_k])$, then $\alpha_k = 1$. If $L_k = (a_k, b_k]$ or $L_k = [a_k, b_k)$, then we set $\mu_k = -{}^1/_2$, $\nu_k = 0$, requiring boundedness of the solution at the point b_k; here $\alpha_k = 0$.

Remark 1. It is best to introduce the singularities of the function $X(z)$ symmetrically also in problems that have some symmetry in the arrangement of the sections L_k and M_j.

Having determined the parameters α_k and knowing the mutual arrangement of the sections L_k and M_j, we find the numbers ω_k by (1.9) and the difference $w_k^- = n_k^+ - n_k^-$, $k = 1, 2, \ldots, l$, by (1.8). Since the numbers $n_k^\pm$ are integers, the differences w_k^- and sums $w_k^+ = n_k^+ + n_k^-$ will be simultaneously even or odd for every k. In addition to the relations mentioned, the numbers w_k^+ and c_k should satisfy conditions (1.6) which according to (1.5) are a system of $l-1$ equations, linearly algebraic in w_k^+ and transcendental in c_k. It is sufficient to introduce just simple poles and zeros $z = c_k$ into (1.5) for the selection of the numbers β_k in the system by setting $|\beta_k| = 1$, $k = 1, 2, \ldots, l-1$.

Let s_k, $k = 1, \ldots, l-1$ be a system of arbitrary continuous curves. Let each curve s_k lie entirely in the upper half-plane $(y > 0)$ or lower half-plane $(y < 0)$ including the appropriate edge L_k^+ and L_k^- of the slit L_k, and have ends at the point a_k and b_k. Then it can be shown that for $w_l^+ = w_l^-$ and an arbitrary distribution of the numbers $\beta_k = \pm 1$ over k and evenness of the numbers w_k^+ system (1.6) has a solution in the form of integers w_k^+ and complex numbers $c_k \in s_k$. In particular, if the line s_k agrees with one of the edges $L_k \pm$, then c_k is a real number.

Remark 2. It is possible to take $l-1$ arbitrary curves instead of $l-1$ curves s_k, $k = 1, \ldots, l-1$ and relationships $w_l^+ = w_l^-$, and to give an arbitrary number w_k^+ of the same evenness as w_k^- for any one k of the l possible ones.

The existence of a continuum of solutions $c_k \in s_k$ has an explicit mechanical meaning: it corresponds to a continual set of the half-plane equilibrium mode for given indices of the singularities μ_k, ν_k, $k = 1, \ldots, l$, and the undetermined parameters $\chi_k, Y_k, X_j', Y_j', \varepsilon^\infty, \sigma_x^\infty$.

A different kind of constraint is imposed below on the total number β of zeros $z = c_k$ (the numbers $\beta_k = -1$). Taking them into account to select some sequence β_k, $k = 1, 2, \ldots, l-1$, and by determining the unknowns w_k^+ and c_k from system (1.6), we obtain the function $X(z)$, which, according to (1.5), has the asymptotic form at infinity

$$X(z) = O(z^{-r}), \quad r = 2l + m - \alpha' - \alpha'' - 2\beta - 1 \tag{1.10}$$

2. We will now construct the general solution of the combined problem (1.3) and (1.4). Setting

$$\Phi(z) = X(z)[\Phi_1(z) + \Phi_2(z)] \tag{2.1}$$

where $\Phi_2(z)$ is a function analytic on M, we obtain a problem on the jump $\Phi_1^+(x) - \Phi_1^-(x) = g(x)[X^+(x)]^{-1}$, $x \in M$, from (1.4), whose solution has the form

$$\Phi_1(z) = \frac{1}{2\pi i} \int_M \frac{g(t)\,dt}{X^+(t)(t-z)}$$

Since $\Phi_1(z) = O(z^{-1}), z \to \infty$, from (2.1) and the condition $\Phi(z) = O(1), \ z \to \infty$, it follows that $r \geqslant -1$, which means that by virtue of (1.10) the number of zeros β is bounded ($E\{x\}$ is the integer part of x)

$$\beta \leqslant E\{{}^1/_2(2l + m - \alpha' - \alpha'')\} \tag{2.2}$$

Let $\operatorname{Im} c_k \neq 0$ for all the zeros $z = c_k$. Then substituting (2.1) into (1.3), we obtain the Dirichlet problem /5/

$$\operatorname{Im}\Phi_2^{\pm}(x) = f_2^{\pm}(x), \quad f_2^{\pm}(x) = f^{\pm}(x)[X^{\pm}(x)]^{-1} - \operatorname{Im}\Phi_1(x), \quad x \in L \tag{2.3}$$

It is natural to assume that the integrable singularities of the function $\Phi(z)$ are radicals by analogy with $X(z)$ (this can be proved rigorously but such a proof is not required when we have a uniqueness theorem for solving problem (1.1) and (1.2)). Then, starting from (2.1) and the asymptotics forms (1.10) and (1.7) of the function $X(z)$ that has radical singularities at all the nodes a_k, b_k except α' of the nodes b_k of the half-open intervals of L_k where it is bounded, the solution of problem (2.3) must be found in the class of functions integrable at the mentioned α' nodes b_k and finite in the remaining $2l - \alpha'$ nodes of the contour L under the additional condition $\Phi_2(z) = O(z^r), \ z \to \infty$.

Taking into account that this solution can have simple poles at β points c_k, we obtain

$$\Phi_2(z) = \frac{Y_0(z)}{2\pi i}\int_L \frac{f_2^+(t) + f_2^-(t)}{Y^+(t)(t-z)}\,dt + \frac{1}{2\pi i}\int_L \frac{f_2^+(t) - f_2^-(t)}{t-z}\,dt + \tag{2.4}$$

$$\frac{1}{2}\sum_{k=1}^{\beta}\left\{\frac{A_k}{z-c_k} + \frac{\bar{A}_k}{z-\bar{c}_k} + Y(z)\left[\frac{A_k}{Y(c_k)(z-c_k)} - \frac{\bar{A}_k}{Y(\bar{c}_k)(z-\bar{c}_k)}\right]\right\} +$$

$$P_r(z) + iQ_s(z)Y_0(z), \quad Y_0(z) = Y(z)\prod_{k=1}^{\alpha'}(z - b_k')^{-1},$$

$$s = r - l + \alpha'$$

$$P_r(z) = C_0 + C_1 z + \ldots + C_r z^r, \quad Q_s(z) = D_0 + D_1 z + \ldots + D_s z^s$$

Here C_k, D_k are arbitrary real and A_k arbitrary complex constants and for simplicity in the writing, the first β numbers c_k are taken as zeros; if the first integral of (2.4) is different from zero, then the condition $X(z)\Phi_2(z) = O(1), \ z \to \infty$, equivalent to the condition $X(z)Y_0(z)z^{-1} = O(1)$, imposes the following constraint on β:

$$\beta \leqslant E\{{}^1/_2(l + m - \alpha')\} \tag{2.5}$$

which is no less stiff than (2.2); the α' nodes b_k are denoted by b_k' at which the function $X(z)$ is bounded, $\nu_k = 0$. In sum, the function $\Phi_2(z)$ contains $N = 2\beta + r + s + 2$ arbitrary real constants. Of these $2(l - \beta - 1)$ constants should go to cancellation of the poles of the function $\Phi(z)$. According to (2.1), the requirement that the functions $\Phi_1(z) + \Phi_2(z)$ vanish with appropriate multiplicity at $l - \beta - 1$ simple complex or double real poles c_k is sufficient for this (if s_k is an edge of L_k, then the poles and zeros $c_k \in s_k$ are doubled at this edge because of the formation of a logarithmic singularity for the function $\psi(z)$ at the point c_k). The number $l + 2m - \alpha' - 2\alpha'' + 2$ of the remaining real constants is independent of β and equals the number of given kinematic and force parameters $\chi_k, Y_k, X_j', Y_j', \varepsilon^\infty, \sigma_x^\infty$ of the initial problem obtained in Sect.1.

Therefore, the N constants (2.4) can be found from the system of N linear algebraic equations; the matrix elements of the system corresponding to the force and kinematic factors are calculated, as usual /4/, by integrating the contact stresses and the boundary displacements. By virtue of the linear independence of the functions (2.4) multiplicity of these N constants and by virtue of the uniqueness of the solution of the elasticity theory problem (1.1) and (1.2), the determinent of the system is different from zero and it has a unique solution. An analogous result is also obtained on combining several stamps into one or for another constraint on their degrees of freedom.

Problem (1.1), (1.2) can be solved in a narrower class of functions, with finite stresses at any N_1 nodes, by starting from (1.2), (2.1), equating the stress intensity factors at these nodes to zero, the obtaining N_1 conditions connecting the given functions (1.1) and all the parameters $a_1, b_1, \ldots, q_m, \chi_k, Y_1', \ldots, X_m', \varepsilon^\infty, \sigma^\infty$, that were independent earlier.

Let us examine modifications of the selection of s_k and β_k. The representation $N = 3l + 2m - \alpha' - 2\alpha'' - 2\beta$ shows that the number of unknowns in (2.4) diminishes as the number of zeros β grows, becoming a minimum for $\beta = l - 1$. However, conditions (2.2) and (2.5) can, on the one hand, hinder an increase in β and on the other, complicate the search for complex zeros, (as compared with the allowable real poles c_k) and the subsequent calculations. If the constraint $\operatorname{Im} c_k \neq 0$ for $\beta_k = -1$ is removed, then the solution of the homogeneous Dirichlet problem (2.3) with given double real poles $c_k \in L_k^{\pm}$ is not expressed in elementary functions (2.4) but in quadratures or is reduced to the solution of two additional systems of equations of the type (1.6); the inhomogeneous problem (2.3) should here be separated into two so that $f^{\pm}(x) \equiv 0, x \in L_k$ for $X^{\pm}(c_k) = 0$.

Following /1/ it is possible to set $\beta_k = -1, c_k \in L_k^{\pm}$ for all k and without reducing the Dirichlet-Riemann problem to the Dirichlet problem, construct the solution in the form of a sum of canonical linearly independent solutions (1.5). In this case the system contains $N = l + 2m - \alpha' - 2\alpha'' + 2$, i.e., the minimum of unknowns, but approximately $^1/_2 N_2$ equations of the type (1.6) must additionally be solved. Therefore, each modification has its advantages and disadvantages in different cases.

3. As an example we consider the Fal'kovich problem /3/ in a more general formulation. Let the half-plane adjoint a flat stamp having two symmetric slip sections $L_1 = [-a, -b)$, $L_2 = (b, a]$ and one adhesion section without tension $M_1 = [-b, b]$. Then

$$\begin{aligned} &a_1 = -a, \quad b_1 = p_1 = -b, \quad a_2 = q_1 = b, \quad b_2 = a, \quad \alpha' = 2, \\ &\alpha'' = 0 \\ &l = 2, \quad m = 1, \quad u_0'(x) = v_0'(x) = \tau_0(x) \equiv 0, \quad X_1' = F\cos\theta, \\ &Y_1' = F\sin\theta, \quad \beta \leqslant 1 \end{aligned} \tag{3.1}$$

Unlike in /3/, here $X_1' \neq 0$ and the absolute stability condition for a crack $\tau_{xy}(\pm b, 0) = 0$, is removed, considerably simplifying the problem.

By virtue of (3.1) we have in the canonical solution (1.5)

$$\begin{aligned} &Z(z) = (z+b)^{-1/2+i\gamma}(z-b)^{-1/2-i\gamma}, \ \arg Z^{\pm}(x) = -s(x) - \\ &\quad \pi m_j^{\pm}, \quad x \in L_j \\ &s(x) = \gamma \ln|(x+b)^{-1}(x-b)|, \quad m_1^{\pm} = 1, m_2^{+} = \delta_1 = \varepsilon_2 = 0, \\ &m_2^{-} = 2, \quad \varepsilon_1 = \delta_2 = -{}^1/_2 \end{aligned} \tag{3.2}$$

Taking account of Remark 1, we set $\mu_1 = \nu_2 = 0$; $\mu_2 = \nu_1 = -{}^1/_2$. Hence and from (1.8), (1.9), (3.2) and taking Remark 2 into account, it follows that

$$\alpha_1 = \alpha_2 = \omega_1 = w_1^{-} = 0, \quad \omega_2 = 1, \quad w_2^{+} = -w_2^{-} = -2 \tag{3.3}$$

Of the two possible modifications $\beta = 0$ and $\beta = 1$ of the solutions (2.1), (1.5), we consider the first. Let $c_2 = c \in L_2^{+}$, $\beta_2 = 1$, $\arg(x-c)^{\pm} = \pi[1 + U(x-c)]$, where $U(x)$ is the Heaviside unit function. Then we have according to (3.1)-(3.3)

$$\begin{aligned} &\psi(z) = \frac{Y(z)}{\pi i}\left[\int\limits_L \frac{s(t)\,dt}{Y^{+}(t)(t-z)} + \frac{\pi}{2}\sum_{j=1}^{2}\int\limits_{L_j}\frac{w_j^{+} + 2U(t-c)}{Y^{+}(t)(t-z)}\,dt\right] + 2\pi \\ &Y(z) = \sqrt{(z^2-a^2)(z^2-b^2)}, \quad Y^{+}(t) = -i(-1)^j Y_1(t), \quad t \in \\ &\quad L_j; \quad Y_1(t) = \sqrt{(a^2-t^2)(t^2-b^2)} \end{aligned} \tag{3.4}$$

Evaluating the first integral in (3.4) /2/, we obtain

$$\begin{aligned} &\psi(z) = \gamma\ln\frac{z-b}{z+b} + \varphi(z), \quad Y_2(t) = \sqrt{(a^2-t^2)(b^2-t^2)} \\ &\varphi(z) = Y(z)\left[\int\limits_{-b}^{b}\frac{\gamma\,dt}{Y_2(t)(t-z)} - \int\limits_{b}^{a}\left(\frac{w_1^{+}}{t+z} - \frac{2}{t-z}\right)\frac{dt}{2Y_1(t)} - \right. \\ &\left.\int\limits_{c}^{a}\frac{dt}{Y_1(t)(t-z)}\right] \end{aligned} \tag{3.5}$$

Substituting (3.1)-(3.5) into (1.5), we obtain

$$X(z) = (z-c)^{-1}(z^2-b^2)e^{i\varphi(z)} \tag{3.6}$$

After analogous substitutions (1.6) takes the form

$$n\int\limits_{b}^{a}\frac{dt}{Y_1(t)} + \int\limits_{c}^{a}\frac{dt}{Y_1(t)} - \gamma\int\limits_{-b}^{b}\frac{dt}{Y_2(t)} = 0, \quad n = -1 - \frac{1}{2}w_1^{+}$$

and can be written in Legendre elliptic integrals of the first kind

$$\begin{aligned} &nK(\lambda') + F(\eta, \lambda') - 2\gamma K(\lambda) = 0, \quad \lambda = a^{-1}b, \quad \lambda' = \sqrt{1-\lambda^2} \\ &\eta = \arcsin[(a^2-c^2)^{1/2}(a^2-b^2)^{-1/2}], \quad \lambda \in (0, 1) \end{aligned} \tag{3.7}$$

where $K(\lambda)$ is the complete and $F(\eta, \lambda)$ the incomplete integral.

Inverting the function $F(\eta, \lambda')$ we obtain an explicit expression for c in terms of n

and λ from (3.7)

$$c = a\sqrt{1-\lambda'^2\,\mathrm{sn}^2(T,\lambda')},\quad T \equiv T(n,\lambda) = 2\gamma K(\lambda) - nK(\lambda') \tag{3.8}$$

Here $\mathrm{sn}(T,\lambda)$ is the Jacobi elliptical sine, and the positive value of the radical is selected from the condition $c \in [b, a]$, $b > 0$.

The inequalities connecting n and λ if n, c are roots of (3.7)

$$0 \leqslant 2\gamma K(\lambda) - nK(\lambda') \leqslant K(\lambda'),\quad n \geqslant 0 \tag{3.9}$$

result from the properties of elliptic functions and Poisson's ratio $K(\lambda) > 0$, $T = F(\eta,\lambda') \geqslant 0$, $F(\eta,\lambda') \leqslant K(\lambda')$, $\gamma > 0$ and (3.8). The function $K(\lambda)$ increases monotonically from $^1/_2\pi$ to ∞ in the interval $0 < \lambda < 1$, the function $K(\lambda')$ decreases monotonically from ∞ to $^1/_2\pi$, consequently, the function $T(n,\lambda)$ also increases monotonically for any $n \geqslant 0$, changing sign for $n \geqslant 1$. It hence follows that for a fixed $n \geqslant 1$ a single root $\lambda = \lambda_n$ exists for the equation $T(n,\lambda) = 0$. Since the inequality (3.9) is satisfied in the interval $[\lambda_n, \lambda_{n+1}]$ for $n \geqslant 0$, where $\lambda_0 = 0$, then for all $\lambda \in (0, 1)$ a unique value $n = E\{2\gamma K(\lambda)K^{-1}(\lambda')\}$, can be determined from (3.9) except for the point $\lambda = \lambda_n$, and then an appropriate c according to (3.8), i.e., roots of Eq.(3.7) can be found.

Remark 3. For all $n \geqslant 1$ Eq.(3.7) has two roots $n, c = a$ and $n-1, c = b$ at the points $\lambda = \lambda_n$.

It follows from the formula for n and the monotonicity of the growth of the function $K(\lambda)$ that the quantity n increases without limit in the interval $0 < \lambda < 1$, running sucessively through the values $0, 1, 2, \ldots$. It follows from the monotonicity of the growth of the elliptic sine in (3.8) in the interval $\lambda_n \leqslant \lambda \leqslant \lambda_{n+1}$ from $\mathrm{sn}(0,\lambda_n') = 0$ to $\mathrm{sn}[K(\lambda_{n+1}'), \lambda_{n+1}'] = 1$ that for each $n \geqslant 0$ the quantity c decreases monotonically from a (for $n \geqslant 1$) to b in $[\lambda_n, \lambda_{n+1}]$.

The general solution (2.1) of problem (1.1), (3.1) has the form $\Phi(z) = X(z)\Phi_2(z)$, the function $X(z)$ is determined in (3.6); according to (1.5), (1.10), (2.2) and (2.4), $r = 2$, $s = 2$, $N = 6$

$$\Phi_2(z) = P_2(z) + iQ_2(z)(z^2 - a^2)^{-1/2}(z^2 - b^2)^{1/2} \tag{3.10}$$

The four conditions at infinity (1.2) and two conditions of boundedness of the solution $\Phi_2^+(c) = 0$, $\Phi_2'^+(c) = 0$ yield the following system of equations in the six arbitrary constants in (3.10)

$$\begin{gathered}C_2 + iD_2 = i\exp(i\zeta + {}^1/_2 in\pi)\,[{}^1/_4\sigma_x^\infty + 2i\mu\varepsilon^\infty(\varkappa+1)^{-1}]\\ (c + i\zeta_1)(C_2 + iD_2) + C_1 + iD_1 = -{}^1/_2 i\pi^{-1}F\exp(i\zeta + i\theta_n)\\ (a^2 - c^2)P_2(c) + Y_1(c)Q_2(c) = 0,\quad \theta_n = \theta + {}^1/_2 n\pi\\ (a^2 - c^2)P_2'(c) + Y_1(c)Q_2'(c) + c(a^2 - b^2)Y_1^{-1}(c)Q_2(c) = 0\\ \zeta = \arcsin\sqrt{\frac{c^2 - b^2}{a^2 - b^2}},\quad \zeta_1 = -\int_{-b}^{b}\frac{\gamma t\,dt}{Y_2(t)} + \int_0^a\frac{nt^2\,dt}{Y_1(t)} - \int_c^a\frac{t^2\,dt}{Y_1(t)}\end{gathered} \tag{3.11}$$

The contact stresses on the slip and adhesion sections have the form $(j = 1, 2)$

$$\begin{gathered}\sigma_y = -\frac{2(-1)^{n_j}}{|x - c|}\left[\frac{(-1)^j P_2(x)\,\mathrm{sh}\,\varphi_1(x)}{\sqrt{x^2 - b^2}} + \frac{Q_2(x)\,\mathrm{ch}\,\varphi_1(x)}{\sqrt{a^2 - x^2}}\right],\quad x \in L_j\\ \sigma_y - i\tau_{xy} = -\frac{(\varkappa+1)e^{i\varphi_2(x)}}{\sqrt{\varkappa}(x - c)}\left[\frac{iP_2(x)}{\sqrt{b^2 - x^2}} - \frac{Q_2(x)}{\sqrt{a^2 - x^2}}\right],\quad x \in [-b, b]\end{gathered} \tag{3.12}$$

$$\varphi_1(x) = (-1)^{j+1}Y_1(x)\varphi_0(x),\quad x \in L_j;\quad \varphi_2(x) = -Y_2(x)\varphi_0(x),$$
$$x \in [-b, b]$$
$$\varphi_0(x) = \int_{-b}^{b}\frac{\gamma\,dt}{Y_2(t)(t - x)} + \int_b^a\frac{n\,dt}{Y_1(t)(t + x)} - \int_c^a\frac{dt}{Y_1(t)(t - x)},\quad x \in [-a, a]$$

where the integrals are evaluated in the Cauchy principal value sense.

We examine the case $\sigma_x^\infty = \varepsilon^\infty = 0$ in greater detail. We find from system (3.11)

$$\begin{gathered}C_0 = [-C_1\Delta_1 + D_1Y(c)\sin^2\zeta]\\ D_0 = -[C_1Y_1(c)\cos^2\zeta + D_1\Delta_2]\\ C_1 = F_*\sin(\zeta + \theta_n),\ D_1 = -F_*\cos(\zeta + \theta_n),\quad C_2 = D_2 = 0\end{gathered} \tag{3.13}$$

$$F_* = {}^1/_2 (\pi c)^{-1} F, \quad \Delta_1 = c^2 \sin^2 \zeta + b^2 \cos^2 \zeta,$$
$$\Delta_2 = c^2 \cos^2 \zeta + a^2 \sin^2 \zeta$$

Asymptotic forms of the stresses at the points of separation of the boundary condition are expressed according to (3.12) and (3.13) by the formulas

$$\sigma_y (x) = K_I (\pm b) [2\pi (-b \pm x)]^{-1/2} + \sigma_0 (\pm b) + O (\sqrt{-b \pm x}), \tag{3.14}$$
$$x \to \pm b \pm 0$$
$$(\sigma_y - i\tau_{xy}) (x) = \sigma_0 (\pm b) - iK_{II} (\pm b) [2\pi (b \mp x)]^{-1/2} +$$
$$O (\sqrt{b \mp x}), \quad x \to \pm b \mp 0$$
$$\sigma_y (x) = K_I (\pm a) [2\pi (a \mp x)]^{-1/2} + O (\sqrt{a \mp x}), \quad x \to \pm a \mp 0$$
$$K_I(\pm b) = (\varkappa + 1)(\varkappa - 1)^{-1} K_{II} (\pm b),$$
$$K_{II} (\pm b) = \pm (\varkappa + 1) cF_* \sqrt{\pi \Delta_1 (\varkappa b)^{-1}} \sin (\delta_1 \mp \theta - {}^1/_2 \pi n)$$
$$K_I (\pm a) = \pm 2F_* \sqrt{\frac{\pi \Delta_2}{a}} \sin (\delta_2 \pm \theta + {}^1/_2 \pi n),$$
$$\delta_1 = \operatorname{arctg} \left(\frac{c}{b} \sqrt{\frac{c^2 - b^2}{a^2 - c^2}} \right), \quad \delta_2 = \operatorname{arctg} \left(\frac{c}{a} \sqrt{\frac{a^2 - c^2}{c^2 - b^2}} \right)$$

($\sigma_0 (\pm b)$ are certain constants).

By virtue of (1.2) and (3.14), the derivative of the normal displacement of the free half-plane boundary at the stamp edges has the form

$$2\mu (\varkappa + 1)^{-1} v' (x) = -{}^1/_2 K_I (\pm a) [2\pi (-a \pm x)]^{-1/2} + \tag{3.15}$$
$$O (\sqrt{-a \pm x}), \quad x \to \pm a \pm 0$$

For the contact stresses on the slip sections to be compressive, it is necessary to satisfy four inequalilities $K_I (\pm a) \leqslant 0$, $K_I (\pm b) \leqslant 0$, which we write in the following form by taking account of (3.14)

$$\sin [\pm \delta_j + (-1)^j (\theta \pm {}^1/_2 \pi n)] \leqslant 0, \quad j = 1, 2; \ n \geqslant 0 \tag{3.16}$$

They generate two sequences of conditions constraining the direction of the forces X_1', Y_1' and the ratio of the lengths of the slip and adhesion sections

$$| \theta - {}^1/_2 \pi | \leqslant \delta_0, \quad n = 1, 5, 9, \ldots; \quad | \theta - {}^3/_2 \pi | \leqslant \delta_0, \quad n = 3, 7, 11, \ldots \tag{3.17}$$

Here $\delta_0 = \delta_1$ for $c \leqslant \sqrt{ab}$, $\delta_0 = \delta_2$, for $c \geqslant \sqrt{ab}$, the first inequality corresponds to separation and the second to stamp impression into the half-plane. Inequalities (3.16), in addition to (3.17), allow solutions at a discrete set of points $\lambda = \lambda_n$ for all even $n \geqslant 0$, but they do not introduce anything new.

Indeed, for $\lambda = \lambda_n$ the equation $T(n, \lambda) = 0$ has two roots with even and odd n according to Remark 3, which determine the identical solution of the contact problem by the uniqueness theorem, and all odd n are already in (3.17).

It follows from (3.17) that the solution of problem (3.1) for $\sigma_x^\infty = \varepsilon_x^\infty = 0$ in intervals $\lambda \in (\lambda_{2s}, \lambda_{2s+1})$, $s = 0, 1, 2, \ldots$, cannot be realized mechanically for any θ. The set of values of λ for which the solution has mechanical meaning in the neighbourhood of the points $\pm a$, $\pm b$ agrees completely with the set of segments $[\lambda_n, \lambda_{n+1}]$ only for $\theta = {}^1/_2 \pi$, $n = 1, 5, \ldots$, and $\theta = {}^3/_2 \pi$, $n = 3, 7, \ldots$, i.e., for $X_1' = 0$. As the force deviates from the normal to either side, each n-th segment is contracted monotonically, being transformed for $| X_1' | = | Y_1' |$ at the point λ_n^* governed by the equation $2\gamma K (\lambda_n^*) - (n + {}^1/_2) K' (\lambda_n^*) = 0$. For $| X_1' | > | Y_1' |$ the problem under consideration has no solution.

The question occurs as to whether conditions (3.17) are sufficient for the inequality $\sigma_y (x, 0) \leqslant 0$ to be satisfied for all $x \in L$. Since sufficiency is strictly well-founded in the problem with one detached section /2/ and the sections L_1, L_2 are small compared M_1 with (according to numerical calculations $\lambda_1 = 0.999$, $\lambda_2 = 1 - 1.26 \cdot 10^{-7}$, the asymptotic form λ_n has the form $\lambda_n = 1 - 8 \exp (-{}^1/_2 \gamma^{-1} \pi n)$), then according to Saint-Venant's principle the influence of the stress $\sigma_y (x, 0)$ for $x \in L_1$ on the value of $\sigma_y (x, 0)$ for $x \in L_2$ is small and the sufficiency of conditions (3.17) obviously holds.

The constraints (3.17) uniquely define the allowable adhesion sections $[-b, b]$ but the whole stamp base $[a_1^*, b_2^*]$ can be broader than the contact section $[-a, a]$ because of the intervals S'.

Indeed, let the parameters θ and λ satisfy condition (3.17). If $\delta_0 = \delta_1$ or inequality (3.17) is strict for $\delta_0 = \delta_2$, then $K_I (\pm a) < 0$ and according to (3.15) the intervals $[a_1^*, b_2^*]$ and $[-a, a]$ should agree, otherwise $v (x) > v (a)$ for $x \in S'$ near $\pm a$. If $\delta_0 = \delta_2$ and equality (3.17) is satisfied, then the intensity factor K_I vanishes at the points $a, -a$, or at both these points in the case of normal forces, the adjacency of the stamp to the half-plane will be smooth there, and under the condition $v (x) \leqslant v (a)$, $x \in S'$ the stamp base can overlap the interval $[-a, a]$.

Following /2/, equations can be written down for exact values of the ultimately large parameters a_1^* and b_2^* but they will differ slightly from the appropriate parameters for the stamp /2/ with one adhesion section $x \in [0, 2b]$ and one slip section $x \in [2b, a+b]$. This also follows from Saint-Venant's principle and the estimate $1 - \lambda_n < 10^{-3}$ for all $n \geqslant 1$. In particular, for a normal separating force $\theta = {}^1/_2\pi$, as in /2/ $a_1^* = -\infty, b_2^* = \infty$ for $\delta_2 = 0$ and $n = 1$. This means that if the sections of L reached a certain threshold value $(ba^{-1} = \lambda_1)$ on increasing, then both detached cracks become globally unstable. In view of the monotonic growth of the factor K_{II} as a function of $a - b$ for $ba^{-1} = \lambda_1$, the process being started of their advancement for a constant force Y_1' results in total separation of the stamp.

During its development on the path to global instability $(ba^{-1} > \lambda_1)$ the crack can, theoretically, arbitrarily pass many deceleration states. Indeed, if $\delta_0 = \delta_2$ or the strict inequality (3.17) holds for $\delta_0 = \delta_1$, then for a sufficiently large quantity F both detached cracks are developed $K_I(\pm b) < 0$. If Eq.(3.17) is satisfied for $\delta_0 = \delta_1$, then the intensity factors K_{II} and K_I vanish at the points b or $-b$ and for $X_1' = 0$ simultaneously at either.

The last case of a normal force is especially interesting since here both cracks become absolutely stable on advancing to points governed by a denumerable set of parameters $\lambda = \lambda_{4s+2}$ for $\theta = {}^1/_2\pi$ and $\lambda = \lambda_{4s+4}$ for $\theta = {}^3/_2\pi$, $s = 0, 1, \ldots$.

This problem was actually examined in /3/, but its mechanical formulation, method of solution and analysis were different. It was assumed that the stamp $[-a, a]$ is impressed in a half-plane by a normal force under conditions of prelimiting friction on $[-b, b]$ and slip outside this interval. The replacement of the prelimiting friction conditions by total adhesion conditions made in /3/ is legitimate in principle, but requires verification that the inequality $|\tau_{xy}| \leqslant -\rho\sigma_y$, $x \in [-b, b]$, $y = 0$, is satisfied after the problem has been solved, where $\rho > 0$ is the coefficient friction.

Although any problem of prelimiting friction has a non-denumerable set of solutions, none of them is realized in this case. This is indicated indirectly even in /3/ itself (the sign of $\sigma_y(x, 0)$ is variable for $x \in [-b, b]$) and in /6/. Tensile stresses are naturally allowable in problem (1.1) when studying detachment at M, and it follows from (3.12) that they always occur in the range $[-b, b]$.

The method of solution /3/ cannot be extended to the general case of the problem (3.1). References to other papers in which is method was used can be found in /6, 7/.

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